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VARIATIONAL ESTIMATES FOR DISPERSION AND ATTENUATION OF WAVES IN RANDOM COMPOSITES—III

FIBRE-REINFORCED MATERIALS

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Abstract—The variational approach developed in Parts I and II is applied to some two-dimensional problems of waves propagating transversely to the fibres in a unidirectional fibre-reinforced composite. Estimates of wave speeds (both bounds and self-consistent estimates) and associated estimates for the attenuation of the mean wave are given for long waves. They are studied in detail for the case of SH waves propagating through materials reinforced by aligned elliptic cylinders. All of the results are sensitive to the statistics of the medium. This is illustrated by considering two distinct pair distribution functions. Simple closed-form expressions are given for the limiting cases of a body weakened by aligned cracks and a body strengthened by aligned rigid plates. By-products of the analysis include a new representation for the two-dimensional dynamic Green's function for an anisotropic medium and an expression in the Rayleigh limit for the scattering cross-section of a single elliptic cylinder in a uniform matrix, both of which may be anisotropic.

1. INTRODUCTION

In a previous paper, Talbot and Willis[1] presented a general study of waves in randomly inhomogeneous, and possibly anisotropic, elastic media, starting from a variational principle[2] that was associated with a set of operator equations for stress and momentum polarizations, first given in[3, 4]. Specifically, for a medium with constitutive relations

$$\sigma = Le,\ p = \rho\dot{u}, \tag{1.1}$$

where σ, e denote stress and strain, p, $\dot{u}$ denote momentum density and particle velocity and L, ρ denote tensor of elastic moduli and mass density, stress and momentum polarizations τ, π are defined so that

$$\tau = (L - L_0)e,\ \pi = (\rho - \rho_0)\dot{u}, \tag{1.2}$$

relative to a comparison body with moduli L_0 and density ρ_0. Then, as shown in[3, 4], τ, π satisfy the operator equations

$$\begin{aligned}(L - L_0)^{-1}\tau + S_x\tau + M_x\pi &= e_0,\\ (\rho - \rho_0)^{-1}\pi + S_t\tau + M_t\pi &= \dot{u}_0,\end{aligned} \tag{1.3}$$

where e_0, $\dot{u}_0$ represent the strain and velocity fields that the given boundary and initial conditions would induce in the comparison body. The operators S_x, etc. are obtained from the Green's function for the comparison body. It was shown in[2] that eqns (1.3) imply a variational principle that reduces to the principle of Hashin and Shtrikman[5] in the static limit, and an extension was noted, which generated the hierarchy of equations which follow from (1.3) where L, ρ are random fields. In particular, for an n-phase medium in which L, ρ take the values L_r, ρ_r in phase r, so that

$$L = \sum_{r=1}^{n} L_r f_r(x),\ \rho = \sum_{r=1}^{n} \rho_r f_r(x), \tag{1.4}$$

where $f_r(x)$ is the indicator for the event "$x \in$ phase r", substitution of the simple trial fields

$$\tau = \sum_{r=1}^{n} \tau_r(x, t) f_r(x),\ \pi = \sum_{r=1}^{n} \pi_r(x, t) f_r(x) \tag{1.5}$$

into the stochastic variational principle generated the equations

$$P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} [S_x(\tau_s P_{sr}) + M_x(\pi_s P_{sr})] = P_r e_0,$$

$$P_r(\rho_r - \rho_0)^{-1}\pi_r + \sum_{s=1}^{n} [S_t(\tau_s P_{sr}) + M_x(\pi_s P_{sr})] = P_r \dot{u}_0, \tag{1.6}$$

in which

$$P_r(x) = \langle f_r(x)\rangle,\ P_{sr}(x', x) = \langle f_s(x')f_r(x)\rangle. \tag{1.7}$$

Equations (1.7) define one- and two-point probabilites for the medium. Relative to the variational principle, eqns (1.6) make optimal use of this information, in the sense that allowance for the configuration in any way more generally than in (1.5) would generate equations involving probabilities of higher order. It is emphasised that (1.5) is not expected to provide the exact field in any particular realization of the composite. It can, however, provide precise bounds which involve only two point statistics, for the Laplace transform of the variational operator, if applied to an initial value problem; this was shown in[2]. It is used, in the present work, to estimate the dispersion relation for the mean wave, in the long-wavelength limit. In this case, its status is less precise but it is not at present clear how it could be improved upon, given only two-point statistics; it does, at least, have the virtue of providing, at lowest order, estimates of Hashin–Shtrikman form for overall moduli, which are "best-possible" in terms of two-point statistics.

Plane-wave solutions of (1.6) were discussed in[1] and explicit formulae were derived, in the form of integrals, for dispersion and attenuation coefficients in the limit of long waves. The integrals were evaluated, and detailed results presented in[6], for a variety of composite media which displayed overall isotropy. The present work is devoted to a corresponding study of waves in composites containing a single family of long aligned fibres, when the waves travel transversely to the fibres. The problem to be considered is thus a two-dimensional analogue of that discussed in[1] and the basic formulation is the same, apart from differences of detail arising from the reduction in dimensionality. A new representation for the two-dimensional Green's function, analogous to the three-dimensional representation given in[3], is developed in Section 2. Then, in Section 3, eqns (1.3) are applied to the study of the scattering of plane waves by a single fibre of elliptical cross-section, in the long-wavelength, or Rayleigh, limit. Section 4 contains the basic equations describing the propagation of waves through a composite, starting from (1.6); the resulting estimates for the dispersion and attenuation in the Rayleigh limit are analogous to those given in[1]. Section 5 specializes to a composite consisting of a matrix containing a single population of identical fibres. This is specialized further in Section 6 to the case in which the fibres have elliptical cross-section. For such a composite, the long-wavelength wave speeds (or equivalently, the overall elastic moduli) can be estimated explicitly, once the pair distribution function $g(x)$ for the fibres is specified. Furthermore, different choices of comparison material yield strict upper and lower bounds, and self-consistent estimates. The bounds might be termed generalized Hashin–Shtrikman bounds, since they are obtainable also from the form of the Hashin-Shtrikman variational principle given by Willis[7], but they depend upon the shape of the fibres and upon the form taken for $g(x)$, reducing to the classical Hashin-Shtrikman bounds[8, 9] in the case of transverse isotropy. The term that defines dispersion contains a complicated integral of $g(x)$ and is not considered in detail. The attenuation term, however, allows for pair correlations through a simple factor Λ, together with a term which has the form of a scattering cross-section.

Finally, several results are presented for the propagation of SH waves. Bounds and self-consistent estimates for long-wavelength wave speeds are displayed graphically, for a variety of fibre cross-sections and for two choices of the pair distribution function. Associated estimates for the attenuation are also given. The formulae reduce significantly in the cases of a matrix reinforced by rigid plates, and a matrix weakened by aligned cracks. These results are summarized, again for two choices of pair distribution function, in an Appendix.

2. PRELIMINARIES

Although the present work follows the general pattern of[1], specialization to two-dimensional problems introduces differences of detail which require discussion.

First, the tensor of elastic moduli L will be taken to depend upon x_1, x_2 only (corresponding to fibres aligned in the 3-direction) and only waves propagating transversely to the 3-direction will be considered. Thus, generally, the displacement u will depend upon x_1, x_2, t only and the strain component $e_{33} = 0$.

As in[1], time-harmonic problems will be considered so that two-dimensional versions of the time-reduced operators S, M are required. They are given by eqns (2.6), (2.7) of[1], with G taking the appropriate two-dimensional form. It is, in fact, possible to deduce the two-dimensional G from (3.11) of[1] by expressing the displacement field generated by a long (but finite) line of body-force as an integral and taking a limit. It is easier, however, to proceed directly, along a route which parallels the one that was followed by Willis[3] in producing the three-dimensional G. Thus, for a uniform comparison material with moduli L_0 and density ρ_0, the two-dimensional G satisfies the equation

$$L_0(\nabla)G + \rho_0\omega^2 G + I\delta(x) = 0, \tag{2.1}$$

where x is taken as the two-dimensional vector (x_1, x_2) and $L_0(\xi)$ is the acoustic tensor for waves propagating in the direction $\xi = (\xi_1, \xi_2)$, with components

$$[L_0(\xi)] = (L_0)_{i\alpha k\beta}\xi_\alpha\xi_\beta, \tag{2.2}$$

the summation convention for Greek suffixes implying summation over the values 1, 2 only. It should be noted that the unit tensor I has components δ_{ij} and, correspondingly, G has the nine components G_{ij}, though these depend only upon x_1, x_2. Now from Gel'fand and Shilov[10], the two-dimensional Dirac delta has the plane-wave representation

$$\delta(x) = -\frac{1}{4\pi^2}\int_{|\xi|=1} (\xi.x)^{-2}\,\mathrm{d}s, \tag{2.3}$$

the integration extending around a unit circle in the ξ-plane. This motivates study of the problem

$$L_0(\nabla)F + \rho_0\omega^2 F = I(\xi.x)^{-2}, \tag{2.4}$$

in which F may be taken as a function of $\xi.x$ only; this reduces (2.4) to the form

$$L_0(\xi)F''(p) + \rho_0\omega^2 F(p) = Ip^{-2}, \tag{2.5}$$

where $p = \xi.x$. The solution of (2.5) can be expressed in terms of the normalized eigenvectors $U^N(\xi)$ and eigenvalues $\rho_0 c_N^2(\xi)$ of $L_0(\xi)$. These satisfy

$$[L_0(\xi) - \rho_0 c_N^2]U^N = 0; \quad U^{NT}U^N = 1, N = 1, 2, 3, \tag{2.6}$$

so that, when ξ is a unit vector, c_N represents the speed with which a plane wave of polarization U^N travels in the direction ξ. The solution F of (2.5) can be given in the form

$$F = \sum_{N=1}^{3} U^N U^{NT} f_N, \tag{2.7}$$

where

$$\rho_0(c_N^2 f_N'' + \omega^2 f_N) = p^{-2}. \tag{2.8}$$

The solution of (2.8) can be found by Fourier transforms. Elementary manipulation followed by the superposition implied by (2.3) to generate G from F then gives the result

$$G(x)=\frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1}\mathrm{d}s\,\frac{U^N U^{NT}}{\rho_0 c_N^2}\left\{\frac{i\pi}{2}\exp(i\omega|\xi.x|/c_N)+\int_0^\infty\frac{\exp(-|\xi.x|z)z\,\mathrm{d}z}{z^2+\omega^2/c_N^2}\right\}. \tag{2.9}$$

In contrast to (3.11) of [1], the representation (2.9) does not decompose directly into a static part ($\omega\to 0$) and a dynamic correction. The operator S, however, does: the components of its kernel follow from (2.9), and (2.6) of [1], as

$$S_{pij}(x)=\frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1}\mathrm{d}s\,\frac{U_p^N U_i^N \xi_j}{\rho_0 c_N^2}\left\{-\frac{1}{\xi.x}-\frac{\pi\omega}{2c_N}\operatorname{sgn}(\xi.x)\exp(i\omega|\xi.x|/c_N)\right.$$
$$\left.+\frac{\omega^2}{c_N^2}\operatorname{sgn}(\xi.x)\int_0^\infty\frac{\exp(-|\xi.x|z)\,\mathrm{d}z}{z^2+\omega^2/c_N^2}\right\}\Bigg|_{(ij)} \tag{2.10}$$

where, as in [1], the suffix (ij) implies symmetrization on these indices. The other operators that are needed follow correspondingly from their definitions. From (2.10) of [1],

$$(S_x)_{pqij}=-\frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1}\mathrm{d}s\,\frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^2}\left\{\frac{1}{(\xi.x)^2}-\frac{i\pi\omega^2}{2c_N^2}\exp(i\omega|\xi.x|/c_N)\right.$$
$$\left.-\frac{\omega^2}{c_N^2}\int_0^\infty\frac{\exp(-|\xi.x|z)z\,\mathrm{d}z}{z^2+\omega^2/c_N^2}\right\}\Bigg|_{(ij)(pq)} \tag{2.11}$$

and, from (2.7) and (2.11) of [1],

$$M_{pi}=i\omega G_{pi}, \tag{2.12}$$

$$(M_x)_{pqi}=i\omega S_{ipq}. \tag{2.13}$$

3. SINGLE SCATTERING IN THE RAYLEIGH LIMIT

Since no two-dimensional problems have yet been solved using the present formulation, it is advantageous to consider first the problem of the scattering of a wave by a single inclusion embedded in an infinite uniform matrix. The moduli and density of the matrix will be taken as L_0, ρ_0 and those of the inclusion as L_1, ρ_1. if the comparison medium is identified with the matrix, the scattering is described by eqns (1.3), where u_0 now denotes the incident wave and the polarizations τ, π are non-zero only over the region occupied by the inclusion. The field u_0 will be taken to have the plane-wave form

$$u_0=m\exp[-i(kn_\alpha x_\alpha+\omega t)], \tag{3.1}$$

so that the unit normal n is orthogonal to the 3-axis and the polarization m and wavenumber k satisfy

$$[k^2L_0(n)-\rho_0\omega^2 I]m=0. \tag{3.2}$$

Once eqns (1.3) are solved, the total field follows as

$$u=u_0+v; \tag{3.3}$$

where the scattered field v is given by

$$v=-S\tau-M\pi. \tag{3.4}$$

Rather than finding v explicitly, attention will be concentrated upon finding the scattering cross-section R of the inclusion. This is defined as the ratio of the mean rate of energy radiation E associated with v to the mean energy flux F associated with u_0. It was shown by Willis[3] that

$$E = \frac{i\omega}{4}\int_D (\tau_{ij}v^*_{i,j} - \tau^*_{ij}v_{i,j})\,dx - \frac{\omega^2}{4}\int_D (\pi_i v^*_i + \pi^*_i v_i)\,dx, \tag{3.5}$$

where D denotes the cross-section of the inclusion, while

$$F = \frac{\rho\omega^3}{2k} m_i m_i, \tag{3.6}$$

the superscript * denoting complex conjugation. The cross-section was called Q in[3] but here it is called R to avoid conflict with the notation in[1, 6].

Equations (1.3) are difficult to solve in general but they simplify drastically in the Rayleigh limit $kd \to 0$, where d denotes a dimension characteristic of the inclusion. In this limit, terms of order ω are neglected so that (1.3) reduce to

$$[(L_1 - L_0)^{-1} + \Gamma^\infty]\tau = -ik\epsilon_0, \tag{3.7}$$

$$(\rho_1 - \rho_0)^{-1}\pi = -i\omega m, \tag{3.8}$$

where Γ^∞ is the static limit of the operator S_x, with kernel

$$\Gamma^\infty_{ijkl} = -\frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} ds\, \frac{\xi_j U_i^N U_k^N \xi_l}{\rho_0 c_N^2}\frac{1}{(\xi.x)^2}\bigg|_{(ij)(kl)}, \tag{3.9}$$

and

$$(\epsilon_0)_{ij} = \frac{1}{2}(m_i n_j + m_j n_i). \tag{3.10}$$

The other operators, M_x, S_t, M_t are at least of order ω and so do not appear. Equation (3.8) gives π immediately, as the extra momentum associated with the inclusion being carried along by the incident wave, while (3.7) characterizes τ as the static polarization induced in an inclusion in a matrix subjected to the uniform remote strain $-ik\epsilon_0$.

A formal solution of (3.7) follows immediately if the inclusion has elliptical cross-section, for then τ may be taken constant over D. Application of Γ^∞ to τ in this case requires evaluation of the integral

$$I(p) = Re\int_D \frac{dx'}{(p - \xi.x' - 0i)^2}, \tag{3.11}$$

when $p = \xi.x$. Inclusion of the term "$0i$" facilitates evaluation of the integral by elementary means. Suppose first that the section D is a circular disc of radius a. Then $I(p)$ can be reduced by transforming to rectangular coordinates (p', q'), where $p' = \xi.x'$, and integrating with respect to q'. This gives

$$I(p) = 2Re\int_{-a}^{a} \frac{(a^2 - p'^2)^{1/2}\,dp'}{(p - p' - 0i)^2} \tag{3.12}$$

which can be evaluated by the method of Muskhelishvili[11] to give

$$I(p) = 2\pi Re\left\{\frac{p - 0i}{((p - 0i)^2 - a^2)^{1/2}} - 1\right\}. \tag{3.13}$$

Thus, in particular, when $p = \xi . x$ and $|x| < a$, $I(p) = -2\pi$ and

$$\Gamma^{x}\tau = P\tau, x \in D, \tag{3.14}$$

where P is a constant tensor with components

$$P_{pqij} = \frac{1}{2\pi} \sum_{N=1}^{3} \int_{|\xi|=1} ds \left. \frac{\xi_q U_p{}^N U_i{}^N \xi_j}{\rho_0 c_N{}^2} \right|_{(pq)(ij)} \tag{3.15}$$

The general case of elliptical $D = \{x: x^T A^T A x < 1\}$ can be reduced to the one just considered by the dual transformations $y = Ax$, $\xi = A^T \zeta$. The result

$$P_{pqij} = \frac{|A|^{-1}}{2\pi} \sum_{N=1}^{3} \int_{|\xi|=1} ds \left. \frac{\xi_q U_p{}^N U_i{}^N \xi_j}{\rho_0 c_N{}^2 \xi^T (A^T A)^{-1} \xi} \right|_{(pq)(ij)} \tag{3.16}$$

was derived by Willis[12] by the alternate method of considering the limit of the corresponding result for a long ellipsoid.

Thus, in the case of elliptical D eqn (3.7) can be replaced by the set of algebraic equations

$$[(L_1 - L_0)^{-1} + P]\tau = -ik\epsilon_0. \tag{3.17}$$

Inverting (3.17), therefore,

$$\tau = -ik[(L_1 - L_0)^{-1} + P]^{-1}\epsilon_0. \tag{3.18}$$

Turning now to the evaluation of the cross-section R, (3.5) reveals the need to evaluate the imaginary part of v and the real part of the associated strain, asymptotically as $\omega \to 0$. Since τ, π are imaginary, the real parts of S, M and the imaginary parts of S_x, M_x are required. The imaginary part of S_x and the real part of M are, respectively, $\omega^2 \Delta S_x$, $\omega \Delta M$, where

$$(\Delta S_x)_{pqij} = \frac{1}{8\pi} \sum_{N=1}^{3} \int_{|\xi|=1} ds \left. \frac{\xi_q U_p{}^N U_i{}^N \xi_j}{\rho_0 c_N{}^4} \right|_{(pq)(ij)}, \tag{3.19}$$

$$\Delta M_{pi} = \frac{1}{8\pi} \sum_{N=1}^{3} \int_{|\xi|=1} ds \frac{U_p{}^N U_i{}^N}{\rho_0 c_N{}^2}, \tag{3.20}$$

asymptotically as $\omega \to 0$, from (2.11), (2.12) and (2.9). The operators S, M_x do not, in fact, contribute, since τ, π are constant and these operators have odd kernels. Hence, to lowest order, for an ellipse with semi-axes a, b,

$$R \sim (\pi ab)^2 \omega^3 [\tau \Delta S_x \tau^* + \pi \Delta M \pi^*]/2F. \tag{3.21}$$

4. MULTIPLE SCATTERING

We now address the two-dimensional analogue of the problem studied in[1]. For an n-phase random medium, substitution of the polarizations (1.5) into the variational principle of Willis[2] and seeking a stationary point leads to the eqns (1.6). Planewave solutions are now sought, by taking

$$\tau_r(x, t) = \tau_r \exp[-i(kn_\alpha x_\alpha + \omega t)], \tag{4.1}$$

$$\pi_r(x, t) = \pi_r \exp[-i(kn_\alpha x_\alpha + \omega t)], \tag{4.2}$$

with $u_0 = 0$ and the probabilities P_r, P_{rs} translation-invariant. The algebraic equations that define the constants τ_r, π_r then take the form

$$P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} \int dx' \Gamma^\infty(x')(P_{sr} - P_sP_r)\, e^{-ikn.x'}\, \tau_s$$
$$= P_r\langle e\rangle - \sum_{s=1}^{n} \int dx'(S_x - \Gamma^\infty)(P_{sr} - P_sP_r)\, e^{-ikn.x'}\tau_s - \sum_{s=1}^{n} \int dx' M_x(P_{sr} - P_sP_r)\, e^{-ikn.x'}\pi_s, \tag{4.3}$$

$$P_r(\rho_r - \rho_0)^{-1}\pi_r = -i\omega P_r\langle u\rangle + i\omega \sum_{s=1}^{n} \int dx'\, S(P_{sr} - P_sP_r)\, e^{ikn.x'}\tau_s$$
$$+ i\omega \sum_{s=1}^{n} \int dx'\, M(P_{sr} - P_sP_r)\, e^{-ikn.x'}\pi_s, \tag{4.4}$$

where

$$\langle u\rangle = -(\tilde{S}\bar{\tau} + \tilde{M}\bar{\pi}), \tag{4.5}$$

$$\langle e\rangle = -(\tilde{S}_x\bar{\tau} + \tilde{M}_x\bar{\pi}), \tag{4.6}$$

$$\bar{\tau} = \sum_{r=1}^{n} P_r\tau_r, \quad \bar{\pi} = \sum_{r=1}^{n} P_r\pi_r \tag{4.7}$$

and $\tilde{S}$ represents the Fourier transform of the operator S, evaluated at (kn, ω), with similar definitions for the other operators; they are given explicitly in[1].

Proceeding now by perturbation theory, as in[1], consider the asymptotic form of (4.3), (4.4) as $\omega \to 0$. The function $P_{sr} - P_sP_r$ is assumed to decay to zero at a rate that defines a characteristic microscopic dimension l. "Small ω" is thus taken to be such that $kl \ll 1$. Retaining just 0(1) terms in (4.3), (4.4) gives

$$P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} A_{rs}\tau_s = P_r\langle e\rangle, \tag{4.8}$$

$$P_r(\rho_r - \rho_0)^{-1}\pi_r = -i\omega P_r\langle u\rangle, \tag{4.9}$$

where

$$A_{rs} = \int dx'\, \Gamma^\infty(x')(P_{rs} - P_rP_s). \tag{4.10}$$

Equations (4.8) have the formal solution

$$\tau_r = \sum_{s=1}^{n} T_{rs}P_s\langle e\rangle, \tag{4.11}$$

giving rise to the approximate constitutive relation

$$\langle\sigma\rangle = \tilde{L}\langle e\rangle, \tag{4.12}$$

where

$$\tilde{L} = L_0 + \sum_{r=1}^{n}\sum_{s=1}^{n} P_rP_sT_{rs}. \tag{4.13}$$

It should be noted, however, that $\langle e\rangle_{33} \equiv 0$. Correspondingly, (4.13) defines all components of $\tilde{L}$ *except* $\tilde{L}_{3333}$. The estimate (4.13) can be obtained directly by employing the static variational principle of Hashin and Shtrikman[5], as outlined in[7].

Now consider the lowest-order perturbations to (4.8), (4.9), by making allowance for the terms that have so far been neglected in (4.3), (4.4). Inspection of the equations shows that these perturbations are of order ω^2. Furthermore, since the term $P_{rs} - P_r P_s$ ensures convergence of the integrals, the perturbations can be found from lowest-order estimates of the relevant kernels. These follow trivially from their definitions, except for the term involving the integral with respect to z. This can be estimated by a device introduced by Fraenkel[13]. The integrand is replaced by the uniform approximation

$$\frac{\exp(-|\xi.x|z)z}{z^2+\omega^2/c_N^2} \sim \frac{z}{z^2+\omega^2/c_N^2} + \frac{\exp(-|\xi.x|z)}{z} - \frac{1}{z}, \tag{4.14}$$

which reduces the integral, following an integration by parts, to the form

$$\int_0^\infty \frac{\exp(-|\xi.x|z)z\,\mathrm{d}z}{z^2+\omega^2/c_N^2} \sim -\left[\gamma + \ln\left(\frac{\omega|\xi.x|}{c_N}\right)\right], \tag{4.15}$$

where γ is Euler's constant. The logarithmic variation with ω is counted as $O(1)$ so far as the asymptotic approximation is concerned.

The perturbed equations now follow in much the same way as in[1]. They may be given in the form

$$P_r(L_r - L_0)^{-1}\tau_r + \sum_{s=1}^{n} A_{rs}\tau_s = P_r\langle e\rangle + \epsilon_r, \tag{4.16}$$

$$P_r(\rho_r - \rho_0)^{-1}\pi_r = -i\omega P_r\langle u\rangle + v_r, \tag{4.17}$$

where

$$\epsilon_r = -\sum_{s=1}^{n} [(k^2 A_{rs}^{(kk)} + \omega^2 A_{rs}^{(\omega\omega)} + i\omega^2 D_{rs})\tau_s + \omega k B_{rs}\pi_s], \tag{4.18}$$

$$v_r = \sum_{s=1}^{n} [\omega k B'_{rs}\tau_s + (\omega^2 C_{rs} + i\omega^2 E_{rs})\pi_s]. \tag{4.19}$$

The constants are as listed.

$$(A_{rs}^{(\omega\omega)})_{pqij} = \frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} \mathrm{d}s\, \frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^4} \Lambda'_{rs}(\xi)\bigg|_{(pq)(ij)}, \tag{4.20}$$

$$(A_{rs}^{(kk)})_{pqij} = \frac{1}{8\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} \mathrm{d}s\, \frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^2} (n.\nabla_\xi)^2 \Lambda'_{rs}(\xi)\bigg|_{(pq)(ij)}, \tag{4.21}$$

$$(B_{rs})_{pqi} = \frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} \mathrm{d}s\, \frac{\xi_q U_p^N U_i^N}{\rho_0 c_N^2} (n.\nabla_\xi)\Lambda'_{rs}(\xi)\bigg|_{(pq)}, \tag{4.22}$$

$$(B'_{rs})_{pij} = (B_{rs})_{ijp}, \tag{4.23}$$

$$(C_{rs})_{pi} = \frac{1}{4\pi^2}\sum_{N=1}^{3}\int_{|\xi|=1} \mathrm{d}s\, \frac{U_p^N U_i^N}{\rho_0 c_N^2} \Lambda'_{rs}(\xi), \tag{4.24}$$

$$(D_{rs})_{pqij} = \frac{1}{8\pi}\sum_{N=1}^{3}\int_{|\xi|=1} \mathrm{d}s\, \frac{\xi_q U_p^N U_i^N \xi_j}{\rho_0 c_N^4} \Lambda_{rs}\bigg|_{(pq)(ij)}, \tag{4.25}$$

$$(E_{rs})_{pi} = \frac{1}{8\pi}\sum_{N=1}^{3}\int_{|\xi|=1} \frac{U_p^N U_i^N}{\rho_0 c_N^2} \Lambda_{rs}, \tag{4.26}$$

$$\Lambda_{rs} = \int (P_{rs} - P_r P_s)\,\mathrm{d}x', \tag{4.27}$$

$$\Lambda'_{rs} = \int \left[\gamma + \ln\left(\frac{\omega|\xi.x'|}{c_N}\right)\right](P_{rs} - P_r P_s)\,\mathrm{d}x'. \tag{4.28}$$

The constant Λ_{rs} is directly analogous to Λ_{rs} as given in[1], the integral now extending over two-dimensional space. Λ'_{rs}, on the other hand, is less simple, the term $\delta(\xi.x)$ that appeared in the three-dimensional problem being replaced by the logarithmic term.

Equations (4.16), 4.17) can now be treated by perturbation theory exactly as in[1]. First, to zeroth order, plane waves

$$\langle u\rangle_N = m_N \exp[-i\omega(k_N n.x + \omega t)] \tag{4.29}$$

are found by solving the equations

$$[k_N{}^2\bar{L}(n) - \bar{\rho}\omega^2 I]m_N = 0, \tag{4.30}$$

where the "overall acoustic tensor" $\bar{L}(n)$ is defined as in (2.2) and

$$\bar{\rho} = \sum_{r=1}^{n} P_r\rho_r \tag{4.31}$$

is the mean density. The associated lowest-order polarizations τ_N, π_N are then defined as the solutions of (4.8), (4.9) with the right sides formed from $\langle u\rangle_N$. Then, finally, the perturbed wavenumber, $k = k_N + \delta k_N$ say, is obtained by solving (4.16), (4.17) by perturbation theory, to give

$$\frac{k^2}{k_N^2} - 1 = Q' + iQ, \tag{4.32}$$

where

$$Q' = \frac{-1}{k^2 m\bar{L}(n)m}\sum_{r=1}^{n}\sum_{r=1}^{n}[\tau_r(k^2A_{rs}^{(kk)} + \omega^2A_{rs}^{(\omega\omega)})\tau_s + 2\omega k\tau_r B_{rs}\pi_s + \omega^2\pi_r C_{rs}\pi_s], \tag{4.33}$$

$$Q = \frac{-\omega^2}{k^2 m\bar{L}(n)m}\sum_{r=1}^{n}\sum_{r=1}^{n}[\tau_r D_{rs}\tau_s + \pi_r E_{rs}\pi_s]. \tag{4.34}$$

In (4.33), (4.34), a suffix N is suppressed: it is to be understood that k, m, take the values k_N, m_N and that τ_r, π_r are the polarizations associated with them through (4.8), (4.9). The only difference between these expressions and the corresponding formulae given in[1] is that Q is now proportional to ω^2 rather than ω^3; and, of course, the constants $A_{rs}^{(kk)}$, etc. are defined differently.

5. A MATRIX-FIBRE COMPOSITE

For a matrix containing a single population of aligned fibres, denote the moduli and density of the fibres by L_1, ρ_1 and those of the matrix by L_2, ρ_2. The formulae of the preceding section simplify somewhat, upon use of the relations

$$\left.\begin{aligned} P_1 + P_2 &= 1, \\ P_{1r} + P_{2r} &= P_r\ (r = 1, 2). \end{aligned}\right\} \tag{5.1}$$

These imply

$$\Lambda'_{11} = -\Lambda'_{12} = \Lambda'_{22}, \tag{5.2}$$

with similar relations for Λ_{rs} and, consequentially,

$$Q' = \frac{-1}{k^2 m\bar{L}(n)m}[(\tau_1 - \tau_2)(k^2 A_{11}^{(kk)} + \omega^2 A_{11}^{(\omega\omega)})(\tau_1 - \tau_2) + 2\omega k(\tau_1 - \tau_2)B_{11}(\pi_1 - \pi_2) + \omega^2(\pi_1 - \pi_2)C_{11}(\pi_1 - \pi_2)]. \quad (5.3)$$

$$Q = \frac{-\omega^2}{k^2 m\bar{L}(n)m}[(\tau_1 - \tau_2)D_{11}(\tau_1 - \tau_2) + (\pi_1 - \pi_2)E_{11}(\pi_1 - \pi_2)]. \quad (5.4)$$

The statistics of a matrix-fibre composite are most likely to be given in terms of the number density n_1 (per unit cross-sectional area) and the pair distribution function $g(x)$ of the fibres. The latter is defined so that the probability density for finding a fibre centered at x, conditional upon there being a fibre centred at the origin, is $n_1 g(x)$. The constant Λ_{11} represents the difference between the area occupied by fibres, conditional upon a fibre covering the origin, and the corresponding area, calculated unconditionally. It follows, therefore, either by direct physical reasoning or by manipulation as given in[6], that

$$\Lambda_{11} = \mathscr{A}P_1\Lambda, \quad (5.5)$$

where $\mathscr{A}$ denotes the cross-sectional area of a fibre, $P_1 = \mathscr{A}n_1$ and

$$\Lambda = 1 + n_1 \int \mathrm{d}x[g(x) - 1]. \quad (5.6)$$

The parameter Λ gives the difference between the expected number of fibres intersecting any large area, conditional upon one fibre having specified location and the corresponding number, calculated unconditionally. Its evaluation causes no special problem. The other parameter, Λ'_{11}, depends upon ξ and is less simply expressed in terms of $g(x)$ than its three-dimensional counterpart.

The relations (5.1) also simplify equations (4.8). As already noted in[12], they allow (4.8) to be given in the form

$$[(L_r - L_0)^{-1} + P']\tau_r - P'\bar{\tau} = \langle e \rangle, \quad (5.7)$$

where

$$P' = A_{11}/P_1P_2. \quad (5.8)$$

It follows from (5.7) that

$$\bar{\tau} = \left\{\sum_{s=1}^{2} P_s[I + (L_s - L_0)P']^{-1}\right\}^{-1} \sum_{s=1}^{2} P_r[I + (L_r - L_0)P']^{-1}(L_r - L_0), \quad (5.9)$$

so that

$$\tilde{L} = \left\{\sum_{s=1}^{2} P_s[I + (L_s - L_0)P']^{-1}\right\}^{-1} \sum_{r=1}^{2} P_r[I + (L_r - L_0)P']^{-1}L_r. \quad (5.10)$$

If L_0 is chosen so that $L_r - L_0$ is either positive or negative definite, then $\tilde{L}$, as given by (5.10), is such that $\tilde{L} - \bar{L}$ is correspondingly definite, where $\bar{L}$ denotes the exact tensor of overall moduli. In the case of (transversely) isotropic statistics, P' can be evaluated explicitly and (5.10) yields the bounds of Hashin[8] and Walpole[9]. A self-consistent estimate for $\bar{L}$ follows by assuming that (5.10) yields $\tilde{L} = \bar{L}$ when L_0 is identified with the overall material, so that $L_0 = \bar{L}$. The resulting equation

$$\tilde{L}(\bar{L}) = \bar{L} \quad (5.11)$$

for $\bar{L}$ is approximate, because the variational estimate (5.10) is based upon a polarization field that is unlikely to be exact for any choice of L_0. Self-consistent" prescriptions of this particular kind have been discussed in[7, 12, 14].

Equations (5.7) also yield an explicit solution for the term $\tau_1 - \tau_2$ that appears in (5.3), (5.4). This is

$$\tau_1 - \tau_2 = [I + (P_2 L_1 + P_1 L_2 - L_0)P']^{-1}(L_1 - L_2)\langle e\rangle. \tag{5.12}$$

6. A COMPOSITE CONTAINING ALIGNED ELLIPTICAL FIBRES

Here we study the implications of our formulae for a matrix containing a set of identical fibres, whose elliptical cross-sections have semi-axes of length a, parallel to Ox_1 and b, parallel to Ox_2. Both fibres and matrix will be taken as isotropic, and two pair distribution functions $g(x)$ will be considered, for both of which the composite overall is orthotropic. The simpler of the two functions $g(x)$ is a two-dimensional version of one that has been considered in[7]: $g(x)$ is taken to depend upon x_1, x_2 only in the combination $(x_1^2/a^2 + x_2^2/b^2)^{1/2}$, so that the composite could be realised (conceptually) by subjecting a transversely isotropic composite with circular fibres to an affine transformation. This $g(x)$ will be referred to as having "elliptical symmetry". The other $g(x)$ is of a type that has been considered by Varadan, Varadan and Pao[15]. $g(x)$ is taken as "transversely isotropic" and so a function of $r = (x_\alpha x_\alpha)^{1/2}$ only, with the restriction $g(x) = 0$ when $r < R$, for some $R > 2a$, to ensure that the fibres do not overlap.

Evaluation of the constant P' requires an expression for $P_{11}(x, 0)$ in terms of n_1 and $g(x)$. This takes the form

$$P_{11}(x_1 0) = n_1^2 \mathscr{A}(x) + n_1 \int_D \mathrm{d}y \int \mathrm{d}y'\, g(x + y - y'), \tag{6.1}$$

where D represents the ellipse $\{y : y_1^2/a^2 + y_2^2/b^2 < 1\}$ and $\mathscr{A}(x)$ represents the area of the intersection of D with an identical ellipse centred at x. The first term on the right side of (6.1) represents the probability that 0 and x lie in the same ellipse and the second is the probability that they lie in different ellipses. It follows now that

$$A_{11} = n_1 \int \Gamma^\infty(x)\mathscr{A}(x)\,\mathrm{d}x + n_1^2 \int \Gamma^\infty(x)\,\mathrm{d}x \int_D \mathrm{d}y \int_D \mathrm{d}y'[g(x + y' - y) - 1], \tag{6.2}$$

having also used $P_1 = \mathscr{A}n_1$, where $\mathscr{A} = \mathscr{A}(0)$ denotes the area of D.

The function $\mathscr{A}(x)$ has elliptical symmetry and this enables the single integral in (6.2) to be evaluated, using the result (3.14). This is independent of the size of the ellipse and it follows that, when applied to any function with elliptical symmetry, Γ^∞ behaves like P times a delta function, with P given by (3.16). The first term on the right side of (6.2) thus reduces to $\mathscr{A}n_1 P = P_1 P$.

The other term depends upon the function g. The integral over x can be written in the alternative form

$$\int \Gamma^\infty(x)[g(x + y' - y) - 1]\,\mathrm{d}x = \int \Gamma^\infty(x - y' + y)[g(x) - 1]\,\mathrm{d}x. \tag{6.3}$$

Suppose first that g has elliptical symmetry. The vector $y' - y$ lies within an ellipse of semi-axes $2a$, $2b$ and the result (3.14) implies that the contribution to the integral for x outside this ellipse is zero. Within the ellipse, $g(x) = 0$ and the integral therefore takes the value $-P$, independently of y, y', so that the integrations over these variables are now accomplished trivially. The result is therefore

$$A_{11} = P_1 P_2 P \tag{6.4}$$

so that

$$P' = P. \tag{6.5}$$

Now consider transversely isotropic $g(x)$. A similar argument shows that the right side of (6.3) reduces to $-P_0$, the value of P appropriate to a circular disc, given by (3.15). Hence, in this case,

$$P' = (P - P_1 P_0)/P_2. \tag{6.6}$$

The tensor P' still depends, of course, upon the choice of comparison medium.

The expression (5.4) for Q can be put in a form in which its relationship to the scattering cross-section R of a single inclusion is displayed, since

$$\left.\begin{aligned} D_{11} &= \mathscr{A}P_1\Delta S_x = \mathscr{A}^2 n_1 \Delta S_x, \\ E_{11} &= \mathscr{A}P_1\Delta M = \mathscr{A}^2 n_1 \Delta M. \end{aligned}\right\} \tag{6.7}$$

Thus, (5.4) can be written

$$Q = n_1 \Lambda \tilde{R}/k, \tag{6.8}$$

where

$$\tilde{R} = -\mathscr{A}^2\omega^3[(\tau_1 - \tau_2)\Delta S_x(\tau_1 - \tau_2) + (\pi_1 - \pi_2)M(\pi_1 - \pi_2)]/2\tilde{F} \tag{6.9}$$

and

$$\tilde{F} = \bar{\rho}\omega^3 m_i m_i/2k = \omega k m\tilde{L}(n)m/2. \tag{6.10}$$

When the dispersion of fibres is dilute, $\tilde{L} \sim L_2$ and it is natural to take $L_0 = L_2$. Then, $\tilde{R}$, $\tilde{F}$ reduce to R, F and, so long as $g(x)$ is such that $\Lambda \to 1$, Q reduces to the form that it has to take if scattering by the fibres is uncorrelated.

7. EXPLICIT FORMULAE FOR SH WAVES

If the tensor L is orthotropic and only the component u_3 of the displacement is different from zero, the only non-trivial parts of the stress-strain relation $\sigma = Le$ are

$$\sigma_{13} = 2le_{13}, \sigma_{23} = 2l'e_{23}, \tag{7.1}$$

where $l = L_{1313}$, $l' = L_{2323}$ and $e_{\alpha 3} = \frac{1}{2}u_{3,\alpha}$. We write, consequently,

$$L = (2l, 2l') \tag{7.2}$$

and

$$L^{-1} = (1/2l, 1/2l'). \tag{7.3}$$

If L is isotropic, with shear modulus μ, then $l = l' = \mu$.

The relevant components of the tensors S_x, M_x, S, M are those with suffixes $\alpha 3\alpha 3$, $\alpha 33$, 33α 33 respectively. These follow from G_{33}, to which only one eigenvector U^N contributes (namely, the unit vector in the 3-direction) and the corresponding wave speed, which we simply call c, is

$$c(\xi) = [(l_0\xi_1^2 + l_0'\xi_2^2)/\rho_0]^{1/2}; \tag{7.4}$$

in (7.4), l_0, l'_0 represent the relevant components of L_0. It follows now, from (3.16), that

$$P_{\alpha 3\beta 3} = \frac{ab}{8\pi}\int_{|\xi|=1} \mathrm{d}s \frac{\xi_\alpha \xi_\beta}{(l_0\xi_1{}^2 + l'_0\xi_2{}^2)(a^2\xi_1{}^2 + b^2\xi_2{}^2)} \tag{7.5}$$

so that, using the notation of (7.2),

$$P = \left(\frac{b}{2m_0(am'_0 + bm_0)}, \frac{a}{2m'_0(am'_0 + bm_0)}\right), \tag{7.6}$$

where $m_0{}^2 = l_0$, $m'^2_0 = l'_0$. The tensor P_0 is obtained from (7.6) by setting $a = b$.

Expressions for overall moduli now follow from (5.10). Again using the notation of (7.2),

$$\tilde{L} = (2\tilde{l}, 2\tilde{l}'), \tag{7.7}$$

where

$$\left.\begin{aligned} \tilde{l} &= \sum_{r=1}^{2} P_r\alpha_r\mu_r \Big/ \sum_{s=1}^{2} P_s\alpha_s, \\ \tilde{l}' &= \sum_{r=1}^{2} P_r\alpha'_r\mu_r \Big/ \sum_{s=1}^{2} P_s\alpha'_s. \end{aligned}\right\} \tag{7.8}$$

The factors α_r, α'_r that appear in (7.8) depend upon the statistics of the medium. If the composite has elliptical symmetry,

$$\left.\begin{aligned} \alpha_r &= 1/[m_0(am'_0 + bm_0) + (\mu_r - l_0)b], \\ \alpha'_r &= 1/[m'_0(am'_0 + bm_0) + (\mu_r - l'_0)a] \end{aligned}\right\} \tag{7.9}$$

while if it has transversely isotropic $g(x)$

$$\left.\begin{aligned} \alpha_r &= 1/[P_2m_0(am'_0 + bm_0)(m'_0 + m_0) + (\mu_r - l_0)[b(m'_0 + m_0) - P_1(am'_0 + bm_0)]], \\ \alpha'_r &= 1/\{P_2m'_0(am'_0 + bm_0)(m'_0 + m_0) + (\mu_r - l'_0)[a(m'_0 + m_0) - P_1(am'_0 + bm_0)]\}. \end{aligned}\right\} \tag{7.10}$$

Bounds for the components $\tilde{l}$, $\tilde{l}'$ of $\tilde{L}$ are obtained by substituting into (7.9) or (7.10) the values $l_0 = l'_0 = \mu_1$ or $l_0 = l'_0 = \mu_2$. Assuming that the fibres are stiffer than the matrix, the former choice yields upper bounds while the latter yields lower bounds. The expressions (7.9), (7.10) simplify somewhat when L_0 is taken isotropic. Self-consistent estimates, however, require the comparison material to have moduli $l_0 = \tilde{l}$, $l'_0 = \tilde{l}'$ so that these involve (7.9), (7.10) in the forms given.

Attenuation of the mean wave in the composite is described by (6.8) which, in turn, requires the evaluation of ΔS_x, ΔM. The relevant components of ΔS_x follow from (3.19) as

$$\Delta S_{x_{\alpha 3 \alpha 3}} = \frac{\rho_0}{32\pi}\int_{|\xi|=1} \mathrm{d}s \frac{\xi_\alpha \xi_\beta}{(l_0\xi_1{}^2 + l'_0\xi_2{}^2)^2}. \tag{7.11}$$

Thus, with the notation of (7.2),

$$\Delta S_x = \frac{\rho_0}{16m_0m'_0}\left(\frac{1}{l_0}, \frac{1}{l'_0}\right). \tag{7.12}$$

The only relevant component of ΔM is ΔM_{33}. From (3.20),

$$\Delta M_{33} = 1/(4m_0m'_0). \tag{7.13}$$

The term $\tau_1 - \tau_2$ follows from (5.12); the substitutions are routine but lead to long formulae which are not displayed.

8. RESULTS AND DISCUSSION

A computer program has been written to evaluate the expressions given in the preceding section, when the fibres and matrix are both isotropic, with shear moduli μ_1, μ_2 respectively. In addition, certain limiting cases can be simplified analytically; these are summarised in the Appendix.

It has been remarked earlier that identifying L_0 with $L_r (r = 1 \text{ or } 2)$ generates estimates $\tilde{L}$ for the exact moduli $\bar{L}$ that are bounds, and that a "self-consistent" estimate is obtained by solving the equation $L_0 = \tilde{L}(L_0)$. Having obtained $\tilde{L}$, the long-wavelength dispersion relation (4.30) shows that an SH wave propagates in the direction $n = (\cos\theta, \sin\theta)$ with speed

$$c = [(\tilde{l}\cos^2\theta + \tilde{l}'\sin^2\theta)/\bar{\rho}]^{1/2}. \tag{8.1}$$

When the fibres have circular cross-section, $\tilde{l} = \tilde{l}'$ and (8.1) is independent of θ. Figure 1 shows plots of c against concentration P_1 for boron fibres with circular cross-section in an aluminium matrix. The three curves correspond to the Hashin–Shtrikman lower bound, the self-consistent estimate and the Hashin–Shtrikman upper bound for $\tilde{l}$. They are normalized to the wave speed of the matrix. The values adopted for the moduli and densities were

$$\mu_1 = 25.0, \quad \mu_2 = 3.87\,\text{GPa}, \quad \rho_1 = 2.53, \quad \rho_2 = 2.72\,\text{g cm}^{-2}.$$

Figure 2 shows corresponding plots of the normalized cross-section R/k^3a^4. These figures demonstrate the sensitivity of the estimates to the choice of comparison medium. At low concentrations, the choice $L_0 = L_2$ is known to produce the exact result and we would speculate that the self-consistent choice $L_0 = \tilde{L}$ is likely to be the most satisfactory over a range of concentrations. The attenuation term Q, as given by (6.8), also requires knowledge of Λ. This depends upon $g(x)$. Plots of Λ against P_1 are shown in Fig. 3 for the "well-stirred" approximation $g(x) \equiv 1$, $|x| > 2a$ and for the Perkus–Yevick $g(x)$, for which Λ has been given by Twersky[16]. The Perkus–Yevick $g(x)$ is not the exact distribution function for a two-dimensional "hard-sphere" ensemble in statistical mechanics. An improved (though still ap-

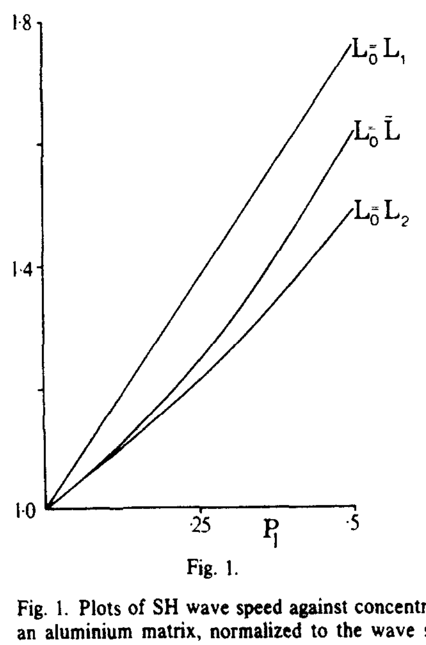

Fig. 1.

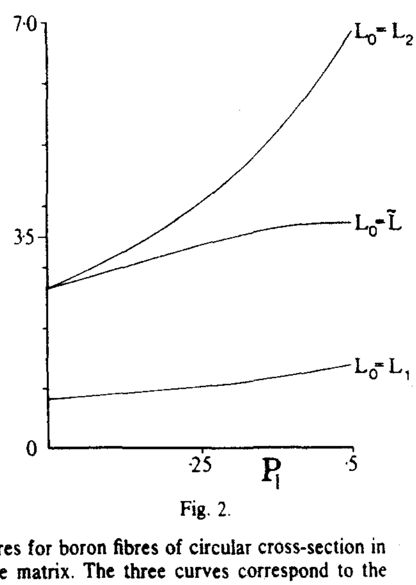

Fig. 2.

Fig. 1. Plots of SH wave speed against concentration of fibres for boron fibres of circular cross-section in an aluminium matrix, normalized to the wave speed of the matrix. The three curves correspond to the Hashin–Shtrikman upper and lower bounds and the self-consistent estimate.

Fig. 2. The normalized cross-section $\tilde{R}/k^3a^4$ plotted against concentration of fibres, for the same composite as in Fig. 1. The three estimates correspond to choosing the properties of the comparison material so as to give the two Hashin–Shtrikman bounds and the self-consistent estimate for the wave speed.

proximate) $g(x)$ has been given by Rowlinson[17]. Since there is no reason in any case to expect a composite to conform to these particular "hard-sphere" statistics, the simple Perkus–Yevick form is adopted for illustration. The main point of interest is that Q is sensitive to $g(x)$. It is noteworthy in particular, that the well-stirred approximation predicts that $\Lambda \leq 0$ for $P_1 \geq 1/4$. This unphysical behaviour has already been remarked upon in[4, 6], in the corresponding three-dimensional problem for which $P_1 = 1/8$ is the critical value. Figure 4 shows plots of the specific attenuation $2P_1\Lambda\bar{R}/(ka)^4$ against P_1, for both functions $g(x)$, with the matrix chosen as comparison material. Plots of specific attenuation are also shown on this figure for waves travelling in the direction $\phi = 0$ in a composite containing fibres of elliptical cross-section with $b/a = 0.6$. They are both for "transversely isotropic" $g(x)$. The one termed "well-stirred" takes $g(x) = 1$, $|x| \geq 2a$ and the one termed P-Y takes the Perkus–Yevick $g(x)$ appropriate to circular cylinders of radius a. Corresponding results for this "well-stirred" approximation were given by Varadan *et al.*[15], for $b/a = 0.6$, 0.8 and 1.0. They employed multiple scattering formalism with the quasicrystalline approximation. Plots of their results for $ka = 0.5$ dip close to zero around $P_1 = 1/4$ but then they rise again. This is presumably an effect of retaining high-order terms in the multipole expansion for the field scattered by any fibre. The present result, however, coincides with that of Varadan *et al.*[15] in the limit $ka \to 0$ and demonstrates that the well-stirred approximation is invalid except at low concentrations.

Figure 5 shows estimates of the wave speed c, as given by (8.1), plotted against θ, for fibres with aspect ratio $b/a = 0.6$, at volume concentration $P_1 = 0.3$, with $g(x)$ transversely isotropic.

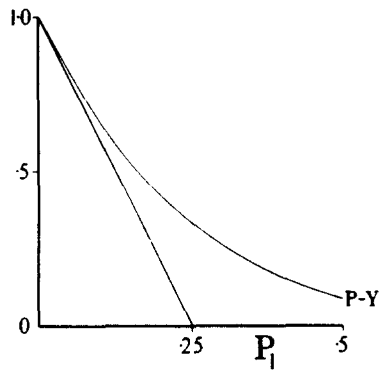

Fig. 3.

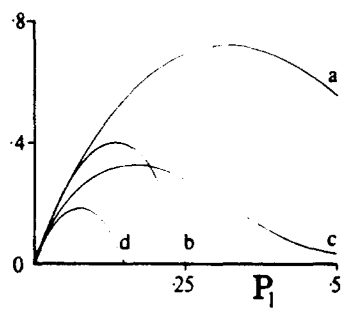

Fig. 4.

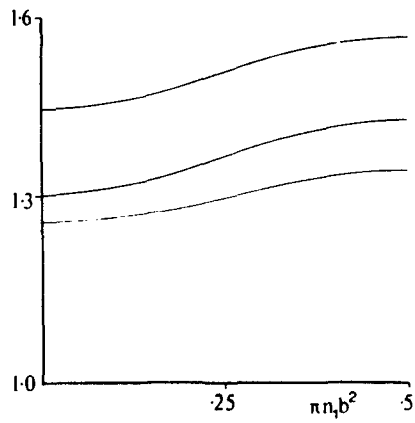

Fig. 5.

Fig. 3. Plots of the structure factor Λ against concentration of fibres, for a matrix containing circular cylindrical fibres, for "well-stirred" and for Percus–Yevick statistics.

Fig. 4. Plots of the specific attenuation $2P_1\Lambda\bar{R}/(ka)^4$ against P_1, for the same composite as in Fig. 1, with the matrix chosen as comparison material, and for a composite containing fibres of elliptical cross-section, with $b/a = 0.6$ and the waves travelling in the direction $\theta = 0$. (a) $b/a = 1$, Percus-Yevic statistics, (b) $b/a = 1$, "well-stirred" statistics, (c) $b/a = 0.6$, Percus-Yevick statistics, (d) $b/a = 0.6$, "well-stirred" statistics.

Fig. 5. Estimates of the wave speed c, plotted against θ, for a $B/A1$ composite with $b/a = 0.6$ and concentration $P_1 = 0.3$. $g(x)$ is taken transversely isotropic, as in[17].

SH waves at any orientation were studied in this type of composite by Varadan and Varadan[17]. We have checked that their estimate for c agrees with ours when $L_0 = L_2$, for the case $\theta = 0$. The present derivation demonstrates that this estimate is, in fact, a lower bound. It has the simple analytic structure given by (5.10) but it is not what might be termed a classical Hashin–Shtrikman bound because of its explicit allowance for two-point statistics. We also have, of course, an upper bound and a self-consistent estimate, which are new.

Finally, Figs. 6–8 show some results for cracks and rigid plates. Figure 6 shows estimates of c against $\pi n_1 b^2$, for cracks with $\theta = 0$ and for rigid plates with $\theta = \pi/2$. The corresponding estimates of R are shown in Fig. 7 for cracks and Fig. 8 for rigid ribbons. For these extreme cases, only the choices $L_0 = L_2$, $\tilde{L}$ are possible, so that only one bound, and a self-consistent estimate, are displayed, for each of the two forms chosen for $g(x)$.

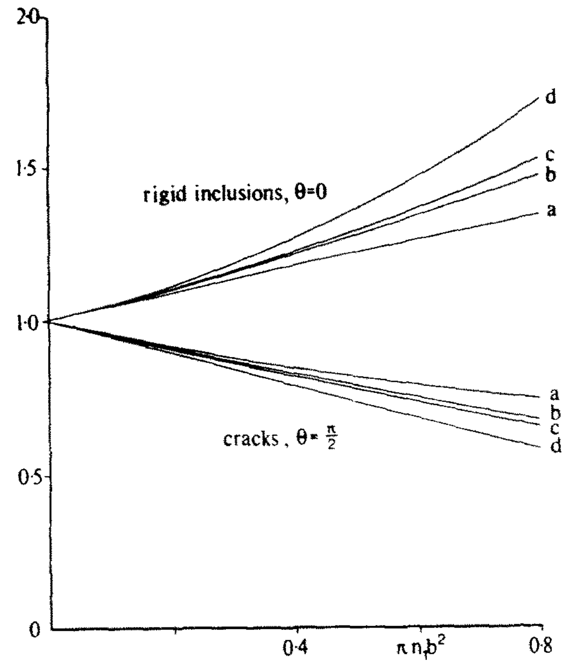

Fig. 6. The wave speed c, plotted against $\pi n_1 b^2$, for waves propagating normally to a set of aligned cracks, and for waves propagating parallel to a set of aligned rigid plates. (a) $L_0 = L_2$, elliptical symmetry, (b) self-consistent, elliptical symmetry, (c) $L_0 = L_2$, transversely isotropic $g(x)$. (d) self-consistent, transversely isotropic $g(x)$.

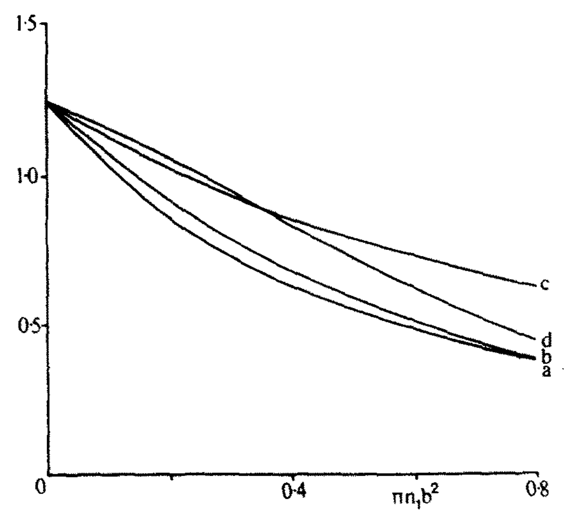

Fig. 7. Estimates of the normalized cross-section $\bar{R}/k^3 b^4$ for waves propagating normally to a set of aligned cracks. Curves are labelled as for Fig. 6.

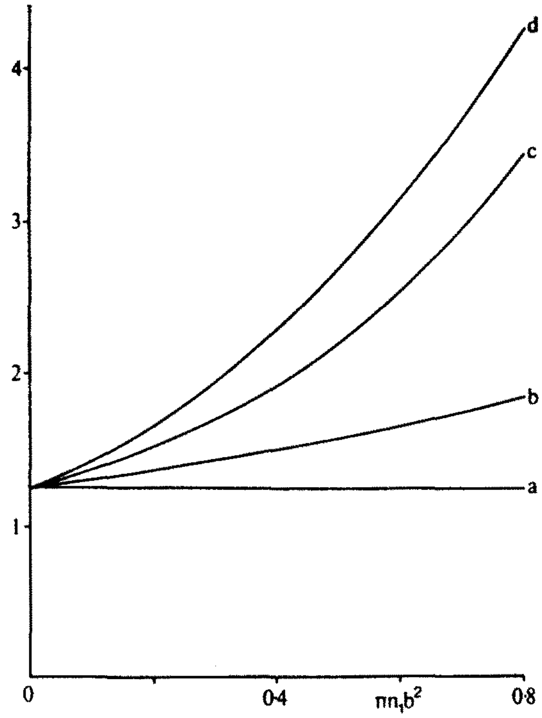

Fig. 8. Estimates of the normalized cross-section $\bar{R}/k^3b^4$ for waves propagating parallel to a set of aligned rigid plates. Curves are labelled as for Fig. 6.

REFERENCES

1. D. R. S. Talbot and J. R. Willis, *Int. J. Solids Structures* **18**, 673 (1982).
2. J. R. Willis, *Wave Motion* **3**, 1 (1981).
3. J. R. Willis, *J. Mech. Phys. Solids* **28**, 287 (1980).
4. J. R. Willis, *J. Mech. Phys. Solids* **28**, 307 (1980).
5. Z. Hashin and S. Shtrikman, *J. Mech. Phys. Solids* **10**, 335 (1962).
6. D. R. S. Talbot and J. R. Willis, *Int. J. Solids Structures* **18**, 685 (1982).
7. J. R. Willis, *J. Mech. Phys. Solids* **25**, 185 (1977).
8. Z. Hashin, *J. Mech. Phys. Solids* **13**, 119 (1965).
9. L. J. Walpole, *J. Mech. Phys. Solids* **17**, 235 (1969).
10. I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol I. Properties and Operations. Academic Press, New York (1964).
11. N. I. Muskhelishvili, *Singular Integral Equations*. Noordhooff, Groningen (1964).
12. J. R. Willis, Mechanics of solids. *The Rodney Hill 60th Anniversary Volume* (Edited by H. G. Hopkins and M. J. Sewell), p. 653. Pergamon Press, Oxford (1982).
13. L. E. Fraenkel, *Proc. Camb. Phil. Soc.* **65**, 233 (1969).
14. J. R. Willis, *Advances in Applied Mechanics* (Edited by C. S. Yih), Vol. 21, p. 1. Academic Press, New York (1981).
15. V. K. Varadan, V. V. Varadan and Y. H. Pao, *J. Acoust. Soc. Amer.* **63**, 1310 (1978).
16. V. Twersky, *J. Acoust. Soc. Amer.* **64**, 1710 (1978).
17. J. S. Rowlinson, *Mol. Phys.* **9**, 217 (1965).
18. V. K. Varadan and V. V. Varadan, *Wave Motion* **1**, 53 (1979).

APPENDIX

This Appendix lists results which are obtained as limiting cases from the general formulae of Section 7.

First, if the fibres have circular cross-section (and fibres and matrix are isotropic), the two models discussed above coincide, so that $P = P_0$. The elastic response of the composite is transversely isotropic so that $\bar{l} = \bar{l}'$. The comparison material is chosen to be transversely isotropic, with $l_0' = l_0 = \mu_0$ and then (7.8), (7.9) give

$$\bar{l} = \frac{\mu_1\mu_2 + (P_1\mu_1 + P_2\mu_2)\mu_0}{P_1\mu_2 + P_2\mu_1 + \mu_0}. \tag{A1}$$

The Hashin–Shtrikman bounds for the exact shear modulus $\bar{l}$ follow by choosing $\mu_0 = \mu_1$ or μ_2 and the self-consistent estimate ($\bar{l} = l_0$) is obtained as the solution of

$$\bar{l}^2 - (P_1 - P_2)(\mu_1 - \mu_2)\bar{l} - \mu_1\mu_2 = 0. \tag{A2}$$

The Hashin–Shtrikman bounds were derived in[8, 9] and the self-consistent eqn (A2) was given in[9].

The "cross-section" $\bar{R}$ for this composite reduces to the form

$$\bar{R} = \mathscr{A}^2k^3\left\{\frac{\rho_0}{2\bar{\rho}}\left(\frac{\mu_1-\mu_2}{c_1\mu_2+c_2\mu_1+\mu_0}\right)^2+\left(\frac{\rho_1-\rho_2}{\bar{\rho}}\right)^2\frac{\mu}{4\mu_0}\right\}, \tag{A3}$$

where $k^2 = \bar{\rho}\omega^2/\bar{\mu}$. The cross-section R of a single fibre is obtained from (A3) by setting $\mu_0 = \bar{\mu} = \mu_2$ and $P_2 = 1$:

$$R = \mathscr{A}^2k^3\left\{\frac{1}{2}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)^2+\frac{1}{4}\left(\frac{\rho_1-\rho_2}{\rho_2}\right)^2\right\}. \tag{A4}$$

The other limiting case that we consider is that of a composite containing plate-like inclusions at fixed number density n_1, with $a/b \to 0$. The volume concentration $P_1 = \pi abn_1$ correspondingly tends to zero and the inclusions have no effect, unless their properties are extreme. Thus, we consider a matrix weakened by aligned cracks ($\mu_1 = \rho_1 = 0$) and a composite containing aligned rigid platelets ($\mu_1/\mu_2 \to \infty$), by first making these substitutions and then taking the limit $a/b \to 0$. The composites so produced have orthotropic elastic properties, which depend upon the form chosen for the function $g(x)$. Before listing the results, we remark that, for a wave with normal $n = (\cos\theta, \sin\theta)$, the two non-trivial components of the mean strain $\langle e\rangle$ are

$$\langle e\rangle = -\frac{1}{2}ik(\cos\theta, \sin\theta) \tag{A5}$$

and the cross-section $\bar{R}$ decomposes into the form

$$\bar{R} = \bar{R}_r\cos^2\theta + \bar{R}_s\sin^2\theta. \tag{A6}$$

These terms come, respectively, from the (13) and (23) components in $(\tau_1-\tau_2)\Delta S_x(\tau_1-\tau_2)$, the term $(\pi_1-\pi_2)\Delta M(\pi_1-\pi_2)$ making no contribution in the limit $a/b \to 0$.

(a) *Aligned cracks, "elliptical" symmetry*

$$\bar{l} = \mu_2 - \frac{\pi n_1 b^2\mu_2^2}{m_0m_0'+\pi n_1b^2\mu_2}, \quad \bar{l}' = \mu_2, \tag{A7}$$

$$\bar{R} = \frac{\pi^2b^4k^3\mu_2^2l_0\cos^2\theta}{8m_0m_0'(m_0m_0'+\pi n_1b^2\mu_2)^2}. \tag{A8}$$

Identifying the comparison material with the matrix, so that $m_0^2 = m_0'^2 = \mu_2$, gives the upper-bound estimate

$$\bar{l} = \mu_2/(1+\pi n_1b^2) \tag{A9}$$

for $\bar{l}$. The self-consistent choice ($l_0 = \bar{l}$, $l_0' = \mu_2$) generates the quadratic equation

$$\beta^2 + \pi n_1b^2\beta - 1 = 0, \tag{A10}$$

where

$$\beta^2 = \bar{l}/\mu_2. \tag{A11}$$

(b) *Aligned cracks, transversely isotropic $g(x)$*
In this case,

$$\bar{l} = \mu_2 - \frac{\pi n_1b^2\mu_2^2(m_0+m_0')}{m_0m_0'(m_0+m_0')+\pi n_1b^2[\mu_2(m_0+m_0')-m_0^3]}, \ \bar{l}' = \mu_2 \tag{A12}$$

$$\bar{R} = \frac{\pi^2b^4k^3\mu_2^2l_0(m_0+m_0')^2\cos^2\theta}{8m_0m_0'\{m_0m_0'(m_0+m_0')+\pi n_1b^2[\mu_2(m_0+m_0')-l_0m_0']\}^2}. \tag{A13}$$

The upper-bound estimate for $\bar{l}$ is

$$\bar{l} = \mu_2(1-\pi n_1b^2/2)/(1+\pi n_1b^2/2) \tag{A14}$$

and the self-consistent equation reduces to

$$(1-\pi n_1b^2)\beta^2 + 2\pi n_1b^2\beta - 1 = 0. \tag{A15}$$

It is, perhaps, worth noting that (A14) is consistent with the low-frequency dispersion relation for the composite under consideration, given by Varadan and Varadan[17]. This can be seen by reducing their complicated eqn (13) to the simpler form

$$K^2/k^2 = (2+\pi a^2n_0)/(2+\pi a^2n_0\cos 2\alpha),$$

using their notation.

(c) *Aligned rigid plates, "elliptical" symmetry*

$$\bar{l} = \mu_2,\ \bar{l}' = \mu_2 + \pi n_1 b^2 m_0 m_0', \tag{A16}$$

$$\bar{R} = \frac{\pi^2 b^4 k^3}{8}\left(\frac{m_0'}{m_0}\right)\sin^2\theta. \tag{A17}$$

Setting $m_0^2 = m_0'^2 = \mu_2$ produces the lower-bound estimate

$$\bar{l}' = \mu_2(1 + \pi n_1 b^2) \tag{A18}$$

and the self-consistent equation, obtained by setting $m_0^2 = \mu_2$, $l_0' = \bar{l}'$ is

$$\beta'^2 - \pi n_1 b^2 \beta' - 1 = 0, \tag{A19}$$

where

$$\beta'^2 = \bar{l}'/\mu_2. \tag{A20}$$

(d) *Aligned rigid plates, transversely isotropic $g(x)$*

$$\bar{l} = \mu_2,\ l' = \mu_2 + \pi n_1 b^2 m_0 m_0'(m_0 + m_0')/[(m_0 + m_0') - \pi n_1 b^2 m_0], \tag{A21}$$

$$\bar{R} = \frac{\pi^2 b^4 k^3 m_0'(m_0 + m_0')^2 \sin^2\theta}{8 m_0(m_0 + m_0' - \pi n_1 b^2 m_0)^2}. \tag{A22}$$

The lower-bound estimate is

$$l' = \mu_2[1 + \pi n_1 b^2/2]/[1 - \pi n_1 b^2/2] \tag{A23}$$

and the self-consistent equation is

$$\beta'^2 - 2\pi n_1 b^2 \beta' - 1 + \pi n_1^2 b^2 = 0. \tag{A24}$$

Cross-sections for a single crack or a single rigid plate follow from the formulae given above by taking $n_1 = 0$ and $l_0 = l_0' = \mu_2$. From (A8) or (A13), the cross-section of a crack is

$$R = \pi^2 b^4 k^3 \cos^2\theta/8 \tag{A25}$$

and, from (A12) or (A22), that of a rigid plate is

$$R = \pi^2 b^4 k^3 \sin^2\theta/8. \tag{A26}$$